Electric Circuit Theory and the Operational Calculus[1]

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CHAPTER IX

The Finite Line with Terminal Impedances

So far in our discussions of wave propagation in lines and wave-filters, we have confined attention to the case where the impressed voltage is applied directly to the infinitely long line. We have found that, by virtue of this restriction, the indicial admittance functions of the important types of transmission systems are rather easily derived and expressible in terms of well known functions, and the essential phenomena of wave propagation clearly exhibited. In practice, however, we are concerned with lines of finite length with the voltage impressed on the line through a terminal impedance Z_1 and the distant end closed by a second terminal impedance Z_2. We now take up the problem presented by such a system.

Let $K=K(p)$ denote the characteristic operational impedance of the line, and $\gamma=\gamma(p)$ the operational propagation constant of the line. We have then

$$\begin{aligned} V&=Ae^{-\gamma x}+Be^{\gamma x}, \\ I&=\frac{1}{K}Ae^{-\gamma x}-\frac{1}{K}Be^{\gamma x}, \end{aligned} \tag{240}$$

where A and B are so far arbitrary constants. To determine these constants we assume an e.m.f. E impressed on the line at $x=0$ through a terminal impedance Z_1 and the line closed at $x=s$ by a second terminal impedance Z_2. At $x=s$ we have therefore

$$Z_2I=V$$

whence from (240)

$$\frac{Z_2}{K}e^{-\gamma s}A-\frac{Z_2}{K}e^{\gamma s}B=Ae^{-\gamma s}+Be^{\gamma s}$$

and

$$B=-\frac{1-\rho_2}{1+\rho_2}e^{-2\gamma s}A \tag{241}$$

where $\rho_2=Z_2/K$.

[1] Concluded from the issue of January, 1926.

At $x=0$ we have

$$V=E-Z_1I$$

whence

$$A+B=E-\frac{Z_1}{K}A+\frac{Z_1}{K}B,$$

$$(1+\rho_1)A+(1-\rho_1)B=E, \tag{242}$$

where $\rho_1=Z_1/K_1$.

From (241) and (242) we get

$$A=\frac{1+\rho_2}{(1+\rho_1)(1+\rho_2)-(1-\rho_1)(1-\rho_2)e^{-2\gamma s}}E$$

$$B=\frac{-(1-\rho_2)e^{-2\gamma s}}{(1+\rho_1)(1+\rho_2)-(1-\rho_1)(1-\rho_2)e^{-2\gamma s}}E$$

and finally

$$I_x=\frac{E}{K+Z_1}\frac{e^{-\gamma x}+\dfrac{1-\rho_2}{1+\rho_2}e^{-\gamma(2s-x)}}{1-\dfrac{1-\rho_1}{1+\rho_1}\dfrac{1-\rho_2}{1+\rho_2}e^{-2\gamma s}}. \tag{243}$$

If we replace E by a unit e.m.f. we get the operational formula for the indicial admittance A_x; thus

$$A_x=\frac{\lambda}{K}\frac{e^{-\gamma x}+\mu_2e^{-\gamma(2s-x)}}{1-\mu_1\mu_2e^{-2\gamma s}}=\frac{1}{Z_x(p)} \tag{244}$$

where

$$\lambda=K/(K+Z_1),$$

$$\mu_1=\frac{1-\rho_1}{1+\rho_1}=\frac{K-Z_1}{K+Z_1},$$

$$\mu_2=\frac{1-\rho_2}{1+\rho_2}=\frac{K-Z_2}{K+Z_2}.$$

K,γ,Z_1,Z_2,μ_1 and μ_2 are, of course, functions of the operator p.

The integral equation corresponding to the operational formula (244) is

$$\frac{1}{pZ_x(p)}=\int_0^\infty e^{-pt}A_x(t)dt. \tag{245}$$

Now by (244) we can expand $1/Z_x(p)$; it is

$$
\begin{aligned}
\frac{1}{Z_x(p)} = \lambda \frac{e^{-\gamma x}}{K} + \lambda\mu_2 \frac{e^{-\gamma(2s-x)}}{K} \\
+\lambda\mu_1\mu_2 \frac{e^{-\gamma(2s+x)}}{K} + \lambda\mu_1\mu_2^2 \frac{e^{-\gamma(4s-x)}}{K} \\
+\lambda\mu_1^2\mu_2^2 \frac{e^{-\gamma(4s+x)}}{K} + \ldots .
\end{aligned}
\tag{246}
$$

Now we observe that $e^{-\gamma x}/K$ is simply the operational formula for the indicial admittance at point x of an infinitely long line with unit e.m.f. impressed directly on the line at $x=0$. This will be denoted by $a_x(t)$. Similarly $e^{-\gamma(2s-x)}/K$ is the operational formula for the indicial admittance at point $(2s-x)$ with unit e.m.f. impressed directly on the line at $x=0$. This will be denoted by $a_{2s-x}(t)$, etc.

Recognition of this fact allows us to derive a formal solution in terms of a series of reflected waves. For let a set of functions $v_o, v_1, v_2, v_3, \ldots$ satisfy and be defined by the operational equations

$$
\begin{aligned}
v_o &= \lambda(p) = \lambda \\
v_1 &= \lambda\mu_2 \\
v_2 &= \lambda\mu_1\mu_2 \\
v_3 &= \lambda\mu_1\mu_2^2, \text{ etc.}
\end{aligned}
\tag{247}
$$

It then follows from the preceding and theorem II that

$$
A_x(t) = \frac{d}{dt}\int_0^t d\tau \left\{ \begin{array}{r} v_o(t-\tau)a_x(\tau) + v_1(t-\tau)a_{2s-x}(\tau) \\ + v_2(t-\tau)a_{2s+x}(\tau) + \ldots \end{array} \right\}. \tag{248}
$$

If, therefore, we know the indicial admittance of the infinitely long line with unit e.m.f. directly applied and if we can solve the operational equations (247), then $A_x(t)$ is given by (248) by integration. This solution may well present formidable difficulty in the way of computation. It is, however, formally straightforward and the numerical computation is entirely possible, the only question being as to whether the importance of the problem justifies the necessary expenditure of time and effort. Without any computations, however, the solution (248) admits of considerable instructive interpretation by inspection. The first term represents the current at point x of an infinitely long line in response to a unit e.m.f. impressed at $x=0$ through an impedance Z_1; $v_o = v_o(t)$ is the corresponding voltage across the line terminals proper. The second term is a reflected wave from the other

terminal due to the terminal irregularity which exists there. The third term is a reflected wave from the sending end terminal, etc. The solution is therefore a wave solution and is expanded in a form which corresponds exactly with the sequence of phenomena, which it represents.

The solution takes a particularly instructive form when $Z_1=k_1K$ and $Z_2=k_2K$ where k_1 and k_2 are numerics. Then

$$\begin{aligned} v_o &= \frac{1}{1+k_1} \\ v_1 &= \frac{1}{1+k_1}\,\frac{1-k_2}{1+k_2} \\ v_2 &= \frac{1}{1+k_1}\,\frac{1-k_2}{1+k_2}\,\frac{1-k_1}{1+k_1}, \text{ etc.} \end{aligned} \tag{249}$$

and

$$A_x(t)=\frac{1}{1+k_1}\left\{\begin{array}{l} a_x(t)+\dfrac{1-k_2}{1+k_2}\,a_{2s-x}(t) \\ +\dfrac{1-k_1}{1+k_1}\,\dfrac{1-k_2}{1+k_2}\,a_{2s+x}(t)+\ \ldots\ . \end{array}\right\}. \tag{250}$$

If $k_1=0$, $k_2=1$ we have the case of the e.m.f. impressed directly on the sending end of the line and the distant end closed through its characteristic impedance; the solution reduces to

$$A_x(t)=a_x(t)$$

as, of course, it should be by definition.

If $k_1=0$ and $k_2=\infty$, we have the case of the line open-circuited at the distant end, and the solution reduces to

$$A_x(t)=\{a_x(t)-a_{2s-x}(t)-a_{2s+x}(t)+a_{4s-x}(t)+\ \ldots\}. \tag{251}$$

Finally, if both k_1 and k_2 are zero, the line is shorted and

$$A_x(t)=\{a_x(t)+a_{2s-x}(t)+a_{2s+x}(t)+a_{4s-x}(t)+\ \ldots\}. \tag{252}$$

The operational equations (247) admit of further interesting and instructive physical interpretation without computation. Consider a circuit consisting of an impedance Z_1 in series with an impedance K. Let a unit e.m.f. be applied to this circuit and let v_o be the re-

sultant voltage across the impedance K. Then, operationally,

$$v_o = \frac{K}{K+Z_1} = \lambda$$

so that v_o, thus defined in physical terms, is the v_o of equations (247).

Now let this voltage be impressed on a circuit consisting of an impedance $2Z_2$ in series with an impedance $K-Z_2$ so that the total impedance is $K+Z_2$. Let the resultant voltage drop across the impedance element $K-Z_2$ be denoted by v_1; then operationally

$$v_1 = \frac{K}{K+Z_1} \cdot \frac{K-Z_2}{K+Z_2} = \lambda\mu_2$$

which agrees with v_1 as given by equation (247).

Similarly if voltage v_1 is applied to a circuit consisting of an impedance $2Z_1$ in series with an impedance $K-Z_1$ and if v_2 denote the voltage drop across impedance $K-Z_1$, then

$$v_2 = \lambda\mu_1\mu_2$$

We can thus see physically what the voltages v_o, v_1, v_2 . . mean in terms of simple circuits consisting of K and Z_1 in series and K and Z_2 in series respectively.

I shall now work out a specific problem exemplifying the preceding theory. The example is made as simple as possible for two reasons. First because its simplicity makes it more instructive than when the phenomena depicted and the essentials of the mathematical methods are obscured by complicated formulas and extensive computations. Secondly while the general method of solution illustrated is thoroughly practical we cannot hope to arrive at the numerical solutions of the complicated problems without a large amount of laborious computations. Problems involving transmission lines with complicated terminal impedances are among the most difficult, as regards actual numerical solution, of any which present themselves in mathematical physics. On the other hand, the formal solution (248) gives at a glance the essential character of the phenomena involved.

The specific problem we shall deal with may be stated as follows: A unit e.m.f. is directly impressed on the terminals of a transmission line of length s, the distant end of which is closed by a condenser C_o. The line is supposed to be non-dissipative, its constants being inductance L and capacity C per unit length. Required the current at any point $x(x<s)$ of the line.

We write $\sqrt{L/C}=k$, $1/\sqrt{LC}=v$: then by virtue of the preceding

analysis of transmission line propagation the indicial admittance a_x of the *infinitely long line* is given by

$$a_x=0, \text{ for } t<x/v,$$

$$=\frac{1}{k}, \text{ for } t\geqq x/v.$$

The operational characteristic impedance is, of course, $k=\sqrt{L/C}$, and the terminal impedances Z_1 and Z_2 are given by

$$Z_1=0,$$

$$Z_2=1/pC_o.$$

Referring now to equation (244) we have:—

$$\lambda=1,\ \mu_1=1,$$

$$\mu_2=\frac{k-1/pC_o}{k+1/pC_o}=\frac{kC_op-1}{kC_op+1}.$$

Consequently, referring to equations (247), we have, operationally,

$$v_o=1$$

$$v_1=v_2=\frac{kC_op-1}{kC_op+1}$$

$$v_3=v_4=\left(\frac{kC_op-1}{kC_op+1}\right)^2$$

$$v_5=v_6=\left(\frac{kC_op-1}{kC_op+1}\right)^3.$$

In order to determine these functions we have therefore to solve the general operational equation

$$V_n=\left(\frac{kC_op-1}{kC_op+1}\right)^n$$

where V_n denotes either v_{2n-1} or v_{2n}.

In order to eliminate the coefficient kC_o, we make use of theorem VIII, and write

$$\phi_n=\left(\frac{p-1}{p+1}\right)^n.$$

In accordance with that theorem

$$V_n(t)=\phi_n(t/kC_o).$$

We therefore start with the operational equation

$$\phi_n = \left(\frac{p-1}{p+1}\right)^n.$$

Now the solution of this operational equation is very easy and can be expressed in a number of ways. We require it expressed in the form most easily computed. The following appears best adapted for our purposes. Consider the auxiliary operational equation:—

$$\sigma_n = \left(\frac{p-2}{p}\right)^n$$

$$= \left(p^n - 2\frac{n}{1!}p^{n-1} + 2^2\,\frac{(n)(n-1)}{2!}\,p^{n-2} + \ldots + (-1)^n 2^n\right)\frac{1}{p^n}.$$

The explicit solution is gotten by replacing $1/p^n$ by $t^n/n!$ and p^n by d^n/dt^n, whence

$$\sigma_n(t) = \left(\frac{d^n}{dt^n} - 2\frac{n}{1!}\frac{d^{n-1}}{dt^{n-1}} + \ldots + (-1)^n 2^n\right)\frac{t^n}{n!},$$

$$= 1 - \frac{n}{1!}\frac{2t}{1!} + \frac{n(n-1)}{2!}\frac{(2t)^2}{2!} + \ldots + (-1)^n\frac{(2t)^n}{n!}.$$

But writing

$$\sigma_n = \left(\frac{p-2}{p}\right)^n = \frac{1}{H(p)}$$

it follows that

$$\phi_n = \left(\frac{p-1}{p+1}\right)^n = \frac{1}{H(p+1)}$$

$$= \frac{p+1}{p}\cdot\frac{p}{p+1}\,\frac{1}{H(p+1)}$$

$$= \left(1+\frac{1}{p}\right)\cdot\frac{p}{p+1}\,\frac{1}{H(p+1)}.$$

Referring now to theorem VII, we see that

$$\phi_n(t) = \left(1 + \int_0^t dt\right)\sigma_n(t)\cdot e^{-t}.$$

Since we have already solved for $\sigma_n(t)$, this determines $\phi_n(t)$ and hence $V_n(t)$. The functions $v_0, v_1, v_2 \ldots$ are therefore determined.

Now refer back to equation (248) giving the required current in terms of $v_o, v_1, v_2 \ldots$ and the admittances $a_x(t), a_{2s-x}(t), \ldots$. It follows at once by substitution of the preceding that

$$A_x(t)=\frac{1}{k}\left\{v_o\left(t-\frac{x}{v}\right)+v_1\left(t-\frac{2s-x}{v}\right)+v_2\left(t-\frac{2s+x}{v}\right)+\ldots\right\}$$

the functions v_o, v_1, v_2 being zero for negative values of the argument. This result may possibly require a little explanation.

Consider the expression

$$\frac{d}{dt}\int_0^t f(t-\tau)\cdot 1(\tau)d\tau$$

where $1(t)$ denotes a function which is zero for $t<t_o$ and unity for $t\geq t_o$. It is evidently identical with the admittance $a_x(t)$ provided the proper value is assigned to t_o.

Now since $1(t)=0$ for $t<t_o$ and unity for $t\geq t_o$, the preceding may be written as zero for $t<t_o$, and

$$\frac{1}{k}\frac{d}{dt}\int_{t_o}^t f(t-\tau)d\tau \qquad \text{for } t\geq t_o$$

which is equal to $f(t-t_o)$.

If we set $x=0$, we get the current entering the line; thus

$$\begin{aligned}A_o(t)&=\frac{1}{k}\left\{v_o(t)+v_1\left(t-\frac{2s}{v}\right)+v_2\left(t-\frac{2s}{v}\right)\right.\\&\qquad\left.+v_3\left(t-\frac{4s}{v}\right)+v_4\left(t-\frac{4s}{v}\right)+\ldots\right\}\\&=\frac{1}{k}\left\{1+2V_1\left(t-\frac{2s}{v}\right)+2V_3\left(t-\frac{4s}{v}\right)\right.\\&\qquad\left.+2V_5\left(t-\frac{6s}{v}\right)+\ldots\right\}.\end{aligned}$$

This has been computed for the case where $\sqrt{sLC_o}=10$ and is shown in Fig. 26. Referring to this figure we see that the current jumps at $t=0$ to the value $\sqrt{C/L}=1/k$, and keeps this constant value for a time interval $2s/v$. At this instant the first reflected wave arrives and the current takes another jump, of $2/k$. Thereafter it begins to decrease very slowly until time $t=4s/v$ at which time it takes another

jump of $2/k$. Thereafter we have a series of jumps of $2/k$ at time intervals $2s/v$, the current decreasing between successive jumps. The smooth curve is the indicial admittance of an oscillation circuit consisting of an inductance sL in series with a capacity C_o. We see therefore, that the current in the line oscillates with discontinuous

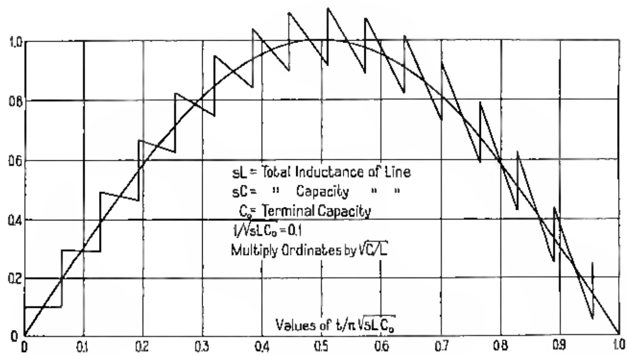

Fig. 26—Current entering non-dissipative line terminated by capacity C_o unit E.M.F. applied to line

jumps about the current in the corresponding oscillation circuit. Since the whole circuit contains no resistance, the oscillations never die away, but continue to oscillate, as shown, about the curve

$$\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{sLC_o}}\right)$$

which is the indicial admittance of the corresponding oscillation circuit.

I shall now discuss a method of solving circuit theory problems, quite generally applicable to complicated networks, and particularly useful in dealing with transmission lines terminated in impedances. I have found it particularly useful in arriving at numerical solutions where other methods prove far more laborious. It is also of mathematical interest, as it applies another type of integral equation to the problems of electric circuit theory.

Suppose that we have a network with two sets of terminals as shown in Fig. 27.[7] Now suppose that terminals 22 are short circuited and a unit e.m.f. inserted between terminals 11. Let the resultant current flowing between terminals 11 be denoted by $S_{11}(t) = S_{11}$ and that

[7] Regarding conventions as to signs, see the Appendix to this chapter.

between terminals 22 by $S_{21}(t)=S_{21}$. S_{11} is the driving point indicial admittance with respect to terminals 11 and S_{21} the transfer indicial admittance of terminals 22 with respect to 11 under short circuit conditions.

Similarly if terminals 11 are shortcircuited and a unit e.m.f. inserted

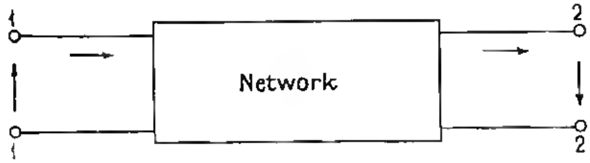

Fig. 27

between terminals 22 the current flowing between terminals 22 is denoted by $S_{22}(t)=S_{22}$ and that flowing between terminals 11 by $S_{12}(t)=S_{12}$. If the network is passive, i.e., contains no internal source of energy, it follows from the reciprocal theorem that $S_{21}=S_{12}$. As far as the two sets of terminals are concerned, the network is completely specified by the indicial admittances $S_{11}, S_{22}, S_{21}=S_{12}$.

Now let a voltage $V_1(t)=V_1$ be inserted between terminals 11, and a voltage $V_2(t)=V_2$ between terminals 22. The current flowing between terminals 11, denoted by I_1 is

$$I_1(t)=\frac{d}{dt}\int_0^t V_1(\tau)S_{11}(t-\tau)d\tau+\frac{d}{dt}\int_0^t V_2(\tau)S_{12}(t-\tau)\,d\tau \qquad (253)$$

while the corresponding current between terminals 22 is

$$I_2(t)=\frac{d}{dt}\int_0^t V_2(\tau)S_{22}(t-\tau)d\tau+\frac{d}{dt}\int_0^t V_1(\tau)S_{21}(t-\tau)\,d\tau \qquad (254)$$

Now consider two networks of indicial admittances S_{11}, S_{22}, $S_{12}=S_{21}$ and $T_{11}, T_{22}, T_{12}=T_{21}$ respectively and let them be connected in tandem as shown in Fig. 28 to form a compound network.

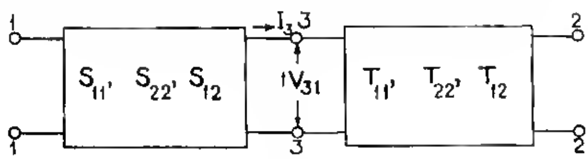

Fig. 28

We require the indicial admittances of the compound network in terms of the indicial admittances of the component networks.

Short circuit terminals 22 of the compound network and insert a

unit e.m.f. between terminals 11. Let $V_{31}(t)$ denote the resultant voltage between terminals 33 measured in the direction of the arrow, and I_3 the current flowing between the networks. We have then the two following expressions for the current I_3.

$$I_3 = S_{21}(t) - \frac{d}{dt}\int_0^t V_{31}(\tau) S_{22}(t-\tau) d\tau \tag{255}$$

and

$$I_3 = \frac{d}{dt}\int_0^t V_{31}(\tau) T_{11}(t-\tau) d\tau. \tag{256}$$

Equating we get

$$\frac{d}{dt}\int^t V_{31}(\tau) [S_{22}(t-\tau) + T_{11}(t-\tau)] d\tau = S_{21}(t). \tag{257}$$

By precisely similar reasoning, if terminals 11 are short circuited and a unit e.m.f. inserted between terminals 22, and the corresponding voltage across terminals 33 denoted by V_{32}, we have [8]

$$\frac{d}{dt}\int_0^t V_{32}(\tau) [S_{22}(t-\tau) + T_{11}(t-\tau)] d\tau = T_{12}(t). \tag{258}$$

Equations (257) and (258) are integral equations of the Poisson type which completely determine V_{31} and V_{32} in terms of the indicial admittances S and T. We shall discuss the solution of these equations presently.

If $U_{11}, U_{22}, U_{21} = U_{12}$ denote the indicial admittances of the compound network we have at once

$$U_{11} = S_{11}(t) - \frac{d}{dt}\int_0^t V_{31}(\tau) S_{12}(t-\tau) d\tau \tag{259}$$

$$U_{22} = T_{22}(t) - \frac{d}{dt}\int_0^t V_{32}(\tau) T_{21}(t-\tau) d\tau \tag{260}$$

and

$$\begin{aligned} U_{21} = U_{12} &= \frac{d}{dt}\int_0^t V_{31}(\tau) T_{21}(t-\tau) d\tau \\ &= \frac{d}{dt}\int_0^t V_{32}(\tau) S_{12}(t-\tau) d\tau. \end{aligned} \tag{261}$$

If, therefore, equations (257) and (258) are solved for V_{31} and V_{32}, the required indicial admittances of the compound network are given

[8] V_{32} being opposite to V_{31} in direction.

by (259), (260) and (261) in terms of the indicial admittances of the component networks.

A simple example will now be worked out illustrating the method of solution just discussed. Suppose that a unit e.m.f. is impressed on a transmission line (infinitely long) of distributed constants $R, L, C,$ through a terminal resistance R_o. Required the terminal line voltage V.

The operational equation of this problem is gotten in the usual manner. The current entering the line is

$$V\sqrt{\frac{Cp}{Lp+R}}.$$

It is also obviously equal to $\frac{1}{R_o}(1-V)$: equating the two expressions, and rearranging we get:—

$$V=\frac{\sqrt{\frac{Lp+R}{Cp}}}{R_o+\sqrt{\frac{Lp+R}{Cp}}}.$$

Writing $R/2L=\rho$ and setting $R_o=\sqrt{L/C}$, this becomes

$$V=\frac{\sqrt{1+2\rho/p}}{1+\sqrt{1+2\rho/p}}. \tag{262}$$

This operational equation can, of course, be solved in a number of ways, though, as a matter of fact, its numerical solution is quite troublesome. This point will be returned to later: we shall first formulate the problem in accordance with the method just discussed.

The indicial admittance of the line is known; it is

$$\sqrt{\frac{C}{L}}e^{-\rho t}I_o(\rho t)=A(t).$$

Consequently the current entering the line is explicitly

$$\frac{d}{dt}\int_0^t V(\tau)A(t-\tau)d\tau.$$

But the current is also equal to $\frac{1}{R_o}(1-V(t))$; equating, we get

$$V(t)=1-R_o\frac{d}{dt}\int_0^t V(\tau)A(t-\tau)d\tau.$$

Performing the indicated differentiations

$$V(t)=1-R_oA(o)\,V(t)-R_o\int_0^t V(\tau)A'(t-\tau)d\tau.$$

Now $A(o)=\sqrt{\frac{C}{L}}$ and

$$A'(t)=\rho e^{-\rho t}(I_1(\rho t)-I_o(\rho t))\sqrt{\frac{C}{L}}$$

and $R_o=\sqrt{L/C}$; therefore the equation becomes

$$V(t)=\frac{1}{2}+\frac{\rho}{2}\int_0^t V(t-\tau)[I_o(\rho\tau)-I_1(\rho\tau)]e^{-\rho\tau}d\tau.$$

As a matter of convenience we change the time scale to ρt, and get

$$V(t)=\frac{1}{2}+\frac{1}{2}\int_0^t V(t-\tau)[I_o(\tau)-I_1(\tau)]e^{-\tau}d\tau \tag{263}$$

$$=\frac{1}{2}-\frac{1}{2}\int_0^t d\tau\ V(t-\tau)\frac{d}{d\tau}e^{-\tau}I_o(\tau),$$

where it is understood that t is actually ρt. This is the integral equation of the problem and is in the canonical form of Poisson's integral equation.

Before solving this equation numerically I shall show how a simple approximate solution is obtainable immediately; an advantage often attaching to this type of integral equation.

The function $\frac{d}{dt}e^{-t}I_o(t)$ is equal to -1 for $t=0$ and converges rapidly to zero. $V(t)$ has, as we know from the operational equation, the initial value 1/2 and the final value 1. Neither function changes sign. It follows from the mean value theorem that the equation can be written as

$$V(t)=\frac{1}{2}-\frac{1}{2}V(t)\int_0^{\alpha t}\frac{d}{dt}e^{-t}I_o(t)dt$$

where $\alpha\leq 1$. Integrating

$$V(t)=\frac{1}{2}-\frac{1}{2}V(t)\ [e^{-\alpha t}I_o(\alpha t)-1]$$

and

$$V(t)=\frac{1}{1+e^{-\alpha t}I_o(\alpha t)}. \tag{264}$$

The correct initial and final values of $V(t)$ result for all final values of $\alpha \leq 1$; so that approximately

$$V(t) = \frac{1}{1+e^{-t}I_o(t)}.$$

This equation, while not exact, except for $t=0$ and t very large, shows faithfully the general character of $V(t)$ and the way it approaches its final value unity. For large values of t

$$e^{-t}I_o(t) = 1/\sqrt{2\pi t}$$

whence

$$V(t) = \frac{1}{1+1/\sqrt{2\pi t}}, \quad t \geq 8. \tag{264-a}$$

Approximations of the foregoing type are not always possible and may not be of sufficient accuracy. I shall therefore give next a method of numerical solution which is generally applicable to integral equations of this type and works quite well in practice. We shall write the integral equation in the more general form

$$u(x) = f(x) + \int_0^x u(x-y)k(y)dy \tag{265}$$

where $f(x)$ and $k(y)$ are known and $u(x)$ unknown. The method depends on the numerical integration of the definite integral. Let us divide the x scale into small intervals d and for convenience write

$$u(nd) = u_n$$
$$f(nd) = f_n$$
$$k(nd) = k_n.$$

Now from the integral equation we have at once

$$u(o) = u_o = f_o,$$
$$u(d) = u_1 = f_1 + \int_0^d u(d-y)k(y)dy.$$

Now if d is taken sufficiently small

$$\int_0^d u(d-y)k(y)dy = \frac{d}{2}[u_1k_o + u_ok_1],$$

whence

$$u_1 = f_1 + \frac{d}{2}[u_1k_o + u_ok_1]$$

and

$$u_1 = \frac{1}{1-k_od/2}[f_1 + u_ok_1d/2]$$

which determines u_1 since u_o is known. Similarly

$$u_2 = f_2 + d\,[\tfrac{1}{2}u_o k_2 + u_1 k_1 + \tfrac{1}{2}u_2 k_o]$$

which determines u_2. Proceeding in the same manner

$$u_3 = f_3 + d\,[\tfrac{1}{2}u_o k_3 + u_1 k_2 + u_2 k_1 + \tfrac{1}{2}u_3 k_o], \text{ etc.}$$

In this way we determine the value of $u(x)$, point by point from the recurrence formula

$$u_n = \frac{f_n + d\,[\tfrac{1}{2}u_o k_n + u_1 k_{n-1} + u_2 k_{n-2} + \ldots + u_{n-1} k_1]}{1 - \tfrac{1}{2}k_o d}. \tag{266}$$

The result of the application of numerical integration, in accordance with formula (266), to the integral equation (263) is shown in Fig. (29). The dotted curve is a plot of the approximate solution as

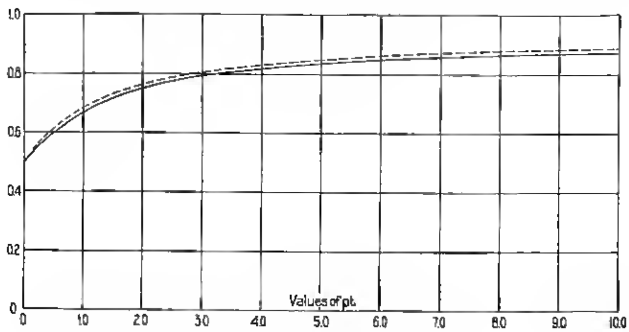

Fig. 29—Line terminal voltage unit E.M.F. impressed on line through resistance $R_0 = \sqrt{L/C}$

given by equation (264), for $\alpha = 1$. We see that the voltage starts with the value 1/2 and slowly reaches its ultimate value, unity, its approach to unity, for large values of t, being in accordance with the formula

$$V(t) = \frac{1}{1 + 1/\sqrt{2\pi t}}.$$

The application of the foregoing method to the transmission line problem proceeds as follows. Let $S_{11}(t)$, $S_{22}(t)$ and $S_{12}(t)$ be the short indicial admittances of the line. $S_{11}(t)$ is the current entering the line (at $x=0$) with unit e.m.f. directly impressed and the distant end short circuited. $S_{12}(t)$ is the current at $x=s$ under the same

circumstances. Consequently from (252)

$$\begin{aligned} S_{11}(t) &= a_o(t)+2a_{2s}(t)+2a_{4s}(t)+\ \ldots \\ S_{12}(t) &= 2\{a_s(t)+a_{3s}(t)+a_{5s}(t)+\ \ldots\}. \end{aligned} \tag{267}$$

S_{22} is clearly equal to S_{11} by symmetry.

Now suppose that an e.m.f. $E=E(t)$ is impressed on the line at $x=0$, $t=0$, through a terminal impedance Z_1, and the distant end $(x=s)$ closed through an impedance Z_2. We suppose these terminal impedances and the actual impressed e.m.f. replaced by the actual line voltages V_1 and V_2, impressed directly on the line at $x=0$ and at $x=s$ are

$$\begin{aligned} I_o(t) &= \frac{d}{dt}\int_0^t S_{11}(t-\tau)\,V_1(\tau)d\tau \\ &\quad -\frac{d}{dt}\int_0^t S_{12}(t-\tau)\,V_2(\tau)d\tau, \end{aligned} \tag{268}$$

$$\begin{aligned} I_s(t) &= -\frac{d}{dt}\int_0^t S_{12}(t-\tau)\,V_1(\tau)d\tau \\ &\quad +\frac{d}{dt}\int_0^t S_{22}(t-\tau)\,V_2(\tau)d\tau. \end{aligned} \tag{269}$$

But the current at $x=s$ is also equal to the current in the terminal impedance Z_2 in response to the terminal voltage V_2: denoting by $\alpha_2(t)$ the indicial admittance of Z_2 it is

$$I_s(t)=\frac{d}{dt}\int_0^t \alpha_2(t-\tau)\,V_2(\tau)d\tau. \tag{270}$$

Similarly the current entering the line at $x=0$ is the current flowing in the terminal impedance Z_1 in response to the e.m.f. $E-V_1$. Denoting by $\alpha_1(t)$ the indicial admittance of Z_1, it is

$$I_o(t)=\frac{d}{dt}\int_0^t \alpha_1(t-\tau)\{E(\tau)-V_1(\tau)\}d\tau. \tag{271}$$

Equating equation (268) and (271) and (269) and (270) we eliminate $I_o(t)$ and $I_s(t)$ and get

$$\begin{aligned} \int_0^t [S_{11}(t-\tau)+\alpha_1(t-\tau)]\,V_1(\tau)d\tau-\int_0^t S_{12}(t-\tau)\,V_2(\tau)d\tau \\ =\int_0^t \alpha_1(t-\tau)E(\tau)d\tau, \end{aligned} \tag{272}$$

$$-\int_0^t S_{12}(t-\tau)\,V_1(\tau)d\tau+\int_0^t [S_{22}(t-\tau)-\alpha_2\,(t-\tau)]\,V_2(\tau)d\tau=0. \tag{273}$$

These two equations are simultaneous integral equations of the Poisson type in V_1 and V_2, which completely determine these voltages provided the admittances and the impressed voltages are known. They therefore represent the application of a new type of integral equation to the problem of electric circuit theory.

The numercial solution of the general case, either by (248) or (272–273) is necessarily laborious when the terminal impedances are complicated and is only justified when the technical importance of the problem is considerable. I wish, however, to emphasize two points in this connection: the numerical solution is always entirely possible and, compared with other and older forms of solution, enormously simpler. One has only to inspect the classical forms of solution of problems of the type to realize the truth of this last statement.

I shall now give two applications of equations (272–273) to specific problems, in one of which an approximate solution of the integral equation can be gotten, and in the other of which numerical integration is applied.

Problem I. Given a non-inductive cable of distributed constants C and R and length s, with unit e.m.f. applied at $x=0$, while at $x=s$ the cable is closed by a condenser C_o. Required the terminal voltage $V(t)$ across the condenser C_o.

We first write down the short-circuit indicial admittances of the cable; from equation (168) of a preceding section and equation (267) they are:—

$$S_{11}(t)=S_{22}(t)$$
$$=\sqrt{\frac{C}{\pi Rt}}\left\{1+2e^{-\frac{4\beta}{t}}+2e^{-\frac{16\beta}{t}}+2e^{-\frac{36\beta}{t}}+\ldots\right\}, \tag{274}$$

$$S_{12}(t)=S_{21}(t)$$
$$=2\sqrt{\frac{C}{\pi Rt}}\left\{e^{-\frac{\beta}{t}}+e^{-\frac{9\beta}{t}}+e^{-\frac{25\beta}{t}}+\ldots\right\}, \tag{275}$$

where $\beta=s^2RC/4$.

Now the current at $x=s$ is equal to

$$S_{12}(t)-\frac{d}{dt}\int_0^t V(\tau)S_{22}(t-\tau)d\tau.$$

It is also the condenser current due to the voltage $V(t)$; that is

$$C_o\frac{d}{dt}V(t).$$

Equating the two expressions and integrating we get

$$C_o V(t) = \int_0^t S_{12}(\tau)d\tau - \int_0^t V(\tau)S_{22}(t-\tau)d\tau \tag{276}$$

which is the integral equation of the problem. In order to get an approximate solution without detailed computation we assume that the cable is long. In this case the leading terms of (274) and (275) are large compared with the terms following: Furthermore $S_{12}(t)$ builds up very slowly while $S_{22}(t)$ is a rapidly varying function. A good approximation therefore results if we take $V(\tau)$ outside the integral sign in (276) and write

$$C_o V(t) = \int_0^t S_{12}(\tau)d\tau - V(t)\int_0^t S_{22}(\tau)d\tau$$

whence

$$V(t) = \frac{1}{C_o} \frac{\int_0^t S_{12}(t)dt}{1+\frac{1}{C_o}\int_0^t S_{22}(t)dt}. \tag{277}$$

This approximation is quite good for long cables and shows the way $V(t)$ builds up quite truthfully. We see that V is initially zero, and builds up ultimately to unity. For large values of t, it becomes

$$V(t) = \frac{\int_0^t S_{12}(t)dt}{\int_0^t S_{22}(t)dt}. \tag{278}$$

This is the approximate formula also for the open circuit voltage, as may be seen by setting $C_o=0$ in (277).

In electric circuit problems, it is often sufficient, as implied above, to know qualitatively the behavior of an electric system without going through the labor of detailed computation. For this purpose the formulation of the problem as a Poisson Integral Equation is particularly well adapted. A simple example will be given, which can be checked from the known solution. Suppose that we require the voltage V at point x of an infinitely long transmission line (L,R,C) in response to a unit e.m.f. impressed at $x=0$. This is, of course, known from formula (211-a): we shall here be concerned, however, with approximate solutions from the Poisson integral equation of the problem.

If $a_x(t)$ denote the indicial admittance of the line at point x, then the current at point x is simply $a_x(t)$, which is given by formula (210-a). But if $V(t)$ is the voltage at point x, the current is also given by

$$\frac{d}{dt}\int_0^t V(\tau)a_o(t-\tau)d\tau.$$

Equating these two expansions, we get the integral equation of the problem

$$\frac{d}{dt}\int_0^t V(\tau)a_o(t-\tau)d\tau = a_x(t).$$

Now if we write $T=\rho t-A$ where $\rho=R/2L$ and $A=\frac{xR}{2}\sqrt{\frac{C}{L}}$, then

$$a_x=\sqrt{\frac{C}{L}}e^{-(T+A)}I_o\sqrt{T(T+2A)}, \quad T\geq 0,$$

and in terms of the relative time T, the integral equation is reducible to

$$\frac{d}{dT}\int_0^T V(T-\tau)e^{-\tau}I_o(\tau)d\tau=e^{-(T+A)}I_o(\sqrt{T(T+2A)}$$

while the exact formula for V is by (211-a)

$$V(T)=e^{-A}+Ae^{-A}\int_0^T\frac{e^{-\tau}I_1\,(\sqrt{\tau(\tau+2A)}}{\sqrt{\tau(\tau+2A)}}d\tau.$$

From the integral equation it is easy to establish superior and inferior limits for $V(T)$; it is

$$V(T)\leq e^{-A}\,\frac{I_o(\sqrt{T(T+2A)}}{I_o(T)}=V_a(T),$$

$$\geq\frac{\int_0^T e^{-T}\,I_o(T)\,V_a(T)dT}{\int_0^T e^{-T}\,I_o(T)dT}=V_b(T).$$

Both formulas give the correct initial and final values of V; namely e^{-A} and unity. Since V lies between V_a and V_b, the mean value $(V_a+V_b)/2$ also has correct initial and final values and should be a better approximation than either. The table given below shows the orders of approximation obtainable from the case where $A=3$. It is evident from this table that the foregoing approximate formulas exhibit the form of $V(T)$ qualitatively in a quite satisfactory manner.

T	V_a	V_b	$\frac{1}{2}(V_a+V_b)$	V
0	0.05	0.05	0.05	0.05
2	0.25	0.12	0.18	0.17
4	0.39	0.19	0.29	0.26
6	0.50	0.23	0.36	0.32
8	0.57	0.27	0.42	0.37
10	0.64	0.31	0.47	0.41
12	0.69	0.34	0.51	0.44
15	0.74	0.38	0.56	0.48
18	0.78	0.41	0.60	0.52

Problem II. Our second illustrative problem may be stated as follows:—A unit e.m.f. is impressed on a transmission line of length s and distributed constants L,R,C. At $x=s$ the line is closed by a resistance R_o in parallel with an inductance L_o. Required the current in the terminal resistance. If $V(t)$ denotes the terminal voltage, the current at $x=s$ is given by

$$S_{12}(t)-\frac{d}{dt}\int_0^t V(\tau)S_{22}(t-\tau)d\tau.$$

It is also equal to the current flowing into the terminal impedance; that is

$$\frac{1}{R_o}V(t)+\frac{1}{L_o}\int_0^t V(\tau)d\tau.$$

Equating and rearranging

$$\left[\frac{1}{R_o}+S_{22}(o)\right]V(t)=S_{12}(t)-\int_0^t V(\tau)\left[\frac{1}{L_o}+S'_{22}(t-\tau)\right]d\tau. \quad (279)$$

Now the short circuit admittance S_{22} and S_{12} are given by formula (210-a) of a preceding chapter, and $S_{22}(o)=\sqrt{C/L}$. In order to apply numerical integration to (279), numerical values must be assigned to the constants. We take

$$R_o=\sqrt{L/C}=1935 \text{ ohms},$$

$$L_o=0.4 \text{ henry},$$

$$\frac{R}{2L}=\rho=292,$$

$$v=1/\sqrt{LC}=1.105\times10^4,$$

$$s=100.$$

The results of the numerical evaluation of equation (279), with these values inserted, is shown in Fig. 30. The voltage is identically zero until $vt=100$; $t=100/v$ is the time of propagation of the line. At that instant it jumps to the value $e^{-\rho t}=e^{-100\rho/v}$ and then begins to die away rapidly due to the draining action of the inductance.

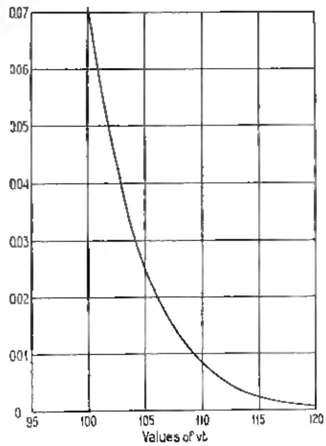

Fig. 30—Voltage across terminal impedance on smooth line

The effect of secondary reflection is insignificant and therefore not shown. The current in the terminal resistance is V/R_0 so that it is given by the same curve.

I have reserved until the last the exposition of the *expansion theorem solution* as applied to transmission lines with terminal impedances, for the reason that it is the least powerful and the most restricted, although most closely resembling the classical form of solution. Furthermore, it does not represent the sequence of physical phenomena, in fact it is not a *wave* solution, but a solution in terms of normal or characteristic vibration. In practical application its usefulness is restricted to the non-inductive cable.

It will be recalled that the expansion theorem solution is formulated as follows:—

If

$$A=1/Z(p)$$

is the operational equation of the problem, then the explicit solution is

$$A(t)=\frac{1}{Z(o)}+\sum_{1}^{n}\frac{e^{p_k t}}{p_k Z'(p_k)}$$

where $p_1, p_2 \ldots$ are the roots of the equation $Z(p)=0$.

Let us apply this formula to the case of a line of length s, with unit e.m.f. directly applied at $x=s$, and line short circuited at $x=s$. Referring to equation (244) and putting $\lambda=\mu_1=\mu_2=1$ we get

$$A_x=\frac{1}{K}\frac{\cosh\gamma(s-x)}{\sinh\gamma s}=\frac{1}{Z_x(p)} \tag{280}$$

as the operational formula of the problem. This can be written as

$$A_x=(Cp+G)\frac{\cosh\gamma(s-x)}{\gamma\sinh\gamma s}=\frac{1}{Z_x(p)} \tag{281}$$

where in the general case,

$$\gamma=\sqrt{(Lp+R)(Cp+G)}. \tag{282}$$

The values of γ for which $Z_x(p)$ vanishes are the roots of the transcendental equation

$$\sinh\gamma s=0$$

excluding zero. These roots are infinite in number: Let γ_m be the m^{th} root; then

$$\gamma_m=i\frac{m\pi}{s},\quad m=1,2,\ \ldots\ \infty. \tag{283}$$

The corresponding values of p_m are then gotten by solving (282) for p and writing $\gamma=\gamma_m$.

The explicit solution of the operational equation (281) is then

$$\begin{aligned}A_x(t)&=\frac{1}{Z_x(o)}+\sum\left(C+\frac{G}{p_m}\right)\frac{\cosh\gamma_m(s-x)}{s\gamma_m\dfrac{d\gamma_m}{dp_m}\cosh\gamma_m s}e^{p_m t}\\&=\frac{1}{Z_x(o)}+\sum(C+G/p_m)\frac{\cosh\gamma_m x}{s\gamma_m\dfrac{d\gamma_m}{dp_m}}e^{p_m t}.\end{aligned} \tag{284}$$

Let us apply this to the non-inductive, non-leaky cable in which $L=G=0$ and $\gamma=\sqrt{RCp}$, so that

$$p_m=\gamma_m^2/RC=-\frac{m^2\pi^2}{s^2RC},$$

and

$$\gamma_m\frac{d\gamma_m}{dp_m}=\frac{RC}{2},$$

Also $Z_x(o)=sR$. We thus get

$$A_x(t)=\frac{1}{sR}+\frac{2}{sR}\sum_{m=1}^{\infty}\cos\frac{m\pi}{s}x.e^{-\frac{m^2\pi^2}{s^2RC}t}. \tag{285}$$

This is a thoroughly practical formula for computation, owing to the rapid convergence of the series. In fact, for this particular line termination chosen, it is probably the simplest and most easily computed form of solution. These advantages depend, however, strictly on two facts. First, the fact that the line is taken as non-inductive and secondly that the terminations chosen are those of a short circuit. In fact, as we shall see, it is only in the case of the non-inductive cable that this type of solution is of any practical value.

There is one other point which should be carefully observed in connection with this solution (285). This is that it is not expressed in terms of a series of direct and reflected waves, corresponding to the sequence of physical phenomena, but in terms of *normal* or *characteristic vibrations*. This point will be returned to later.

Let us now attempt to apply this type of solution to the transmission line, L,R,C,G. Writing

$$\rho=\frac{R}{2L}+\frac{G}{2C},$$

$$\sigma=\frac{R}{2L}-\frac{G}{2C},$$

$$v=1/\sqrt{LC}.$$

We have

$$\gamma^2=\frac{1}{v^2}[(p+\rho)^2-\sigma^2]$$

whence

$$\begin{aligned}p_m&=-\rho\pm v\sqrt{\gamma_m^2+\frac{\sigma^2}{v^2}}\\&=-\rho\pm iv\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\sigma^2}{v^2}},\qquad m=1,2,\ldots\end{aligned}$$

$$\begin{aligned}\gamma_m\frac{d\gamma_m}{dp_m}&=\frac{1}{v^2}(p_m+\rho)\\&=\pm\frac{i}{v}\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\sigma^2}{v^2}}.\end{aligned}$$

Setting $G=0$ for simplicity and substituting in (284) we get, after easy simplifications,

$$A_x(t)=\frac{1}{sR}+\frac{2vC}{s}\sum\frac{\cos\left(\frac{m\pi}{s}x\right)}{\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}}\sin\left(vt\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}\right)e^{-\rho t}. \quad (286)$$

If we write

$$\sqrt{\left(\frac{m\pi}{s}\right)^2-\frac{\rho^2}{v^2}}=\mu_m\frac{m\pi}{s}$$

(286) can be written as

$$A_x(t)=\frac{1}{sR}+\frac{vC}{s}\sum\frac{e^{-\rho t}}{\mu_m\frac{m\pi}{s}}\left\{\sin\frac{m\pi}{s}(\mu_m vt-x)+\sin\frac{m\pi}{s}(\mu_m vt+x)\right\}. \quad (287)$$

This type of solution is often referred to as a *wave* solution and the component terms of the series regarded as travelling waves. As a matter of fact it is a solution in terms of normal or characteristic vibrations, each of which is to be regarded as instantaneously produced at time $t=0$. The solution in terms of true waves has been fully discussed in the preceding.

Formula (287) is practically useless for computation on account of the slow convergence of the series (the series are only conditionally convergent), and cannot be interpreted to bring out the existence of the actual direct and reflected waves and the physical character of the phenomena it formulates. In fact, as stated above, this form of solution is useful only in connection with the non-inductive cable.

In the cases considered above we have taken the simplest possible terminations—these of short circuits in which case the roots of $Z(p)$ are easily evaluated. If, however, the line is closed by arbitrary impedances, the case is quite different, and the location of the roots becomes, except for simple impedances, and then only in the case of the non-inductive cable, practically impossible. While, therefore, the expansion theorem solution can be formally written down, its actual numerical evaluation is a practical impossibility, except in a few cases. For this reason it will not be considered further here.

The physically artificial character of the expansion solution, as applied to transmission lines, may be seen from the following considerations. When a wave is sent into the line, for a finite time equal to the time of the propagation of the line, it is independent of the character of the distant termination. Yet in the expansion solution every term involves and is dependent upon the impedance constitut-

ing the distant termination. Evidently, from physical considerations, the series of component vibrations making up the complete solution must therefore so combine as to annihilate the effect of the distant termination for a finite time. The solution is, therefore, mathematically correct but physically artificial.

Note on Integral Equations.

An integral equation is defined as an equation in which the unknown function occurs under a sign of integration; the process of determining the unknown function is called solving the equation.

Integral equations are of great importance in mathematical physics and in recent years very considerable work has been done on them from the standpoint of pure analysis.

The types of integral equations with which we are concerned in the present work are *Laplace's Equation*

$$F(p) = \int_0^\infty e^{-pt} f(t)dt$$

and *Poisson's Equation*

$$\phi(x) = f(x) + \int_0^x \phi(y)K(x-y)dy.$$

But little work has been done on Laplace's Equation from the standpoint of pure analysis; its most extensive and useful applications appear to be in connection with the Operational Calculus. Practical methods of solution are extensively discussed in the text.

We shall now briefly discuss the solution of Poisson's Equation.

The formal series solution, which is absolutely convergent, is obtained by successive substitution. Thus suppose we write

$$\phi(x) = \phi_o(x) + \phi_1(x) + \phi_2(x) + \ldots .$$

and define the terms of the series in accordance with the scheme

$$\phi_o(x) = f(x),$$

$$\phi_1(x) = \int_0^x \phi_o(y)K(x-y)dy,$$

$$\phi_2(x) = \int_0^x \phi_1(y)K(x-y)dy, \text{ etc.},$$

the resulting series satisfies the integral equation and is absolutely convergent. It is, however, practically useless for computation or interpretation.

A power series solution, when it exists, can be gotten by repeated differentiation; thus

$$\phi(o)=f(o),$$

$$\phi'(x)=f'(x)+\phi(o)K(x)+\int_0^x \phi'(x-y)K(y)dy,$$

$$\phi'(o)=f'(o)+\phi(o)K(o)$$

In this way all the derivatives at $x=0$ are calculable; let them be denoted by $\phi_o, \phi_1, \phi_2 \ldots$ Then

$$\phi(t)=\phi_o+\phi_1\frac{x}{1!}+\phi_2\frac{x^2}{2!}+\ldots.$$

This form of solution, also, is of limited practical usefulness, except for small values of x.

A number of mathematicians, including Wittaker and Bateman, have studied the question of numerical solution and suggested other processes. After quite extensive study of the question, however, the writer is of the opinion that point-by-point numerical integration like that discussed in the text is, in general, the most practical, rapid and accurate method of numerical solution. This judgment is confirmed by G. Prasad who, in a paper on the Numerical Solution of Integral Equations delivered before the International Mathematical Congress (Toronto, 1924), discusses the whole question and arrives at the same conclusion.

In the text, numerical integration is carried out in accordance with Simpson's Rule. It is possible, of course, to employ more complicated and refined formulas for approximate quadrature. It is the writer's opinion that this is hardly justified in practical problems and that the required accuracy is more simply obtained by employing smaller intervals.

Appendix to Chapter IX. Note on Conventions as to Signs in Networks

In the network shown on page 196 the arrows indicate the directions chosen as positive in the network itself, quite regardless of the presence of any e.m.fs. and currents.

The sign attributed to a current, an e.m.f., or a voltage is positive if the current, e.m.f., or voltage is in the positive direction; otherwise the sign is negative.

Stated more fully:

A current at a specific point (at a specific instant of time) is posi-

tive if it is flowing in the positive direction; negative if flowing in the negative direction.

An e.m.f. or a voltage between two points is positive if the potential increases in the positive direction between the two points; negative if the potential increases in the negative direction. (It may be noted that this convention makes the sign of a voltage the same as the sign of that e.m.f. which could be inserted between the two points without producing any effects in the network.)

CHAPTER X

Introduction to the Theory of Variable Electric Circuits[9]

In the preceding chapters it has everywhere been assumed that the networks are *invariable*: that is to say, that the constants and connections of the network do not vary or change with time. In many important technical problems, however, we wish to know, not merely what happens when an electromotive force is applied to an invariable network, but the effect of suddenly changing a circuit constant or of introducing a variable circuit element. In the present chapter we shall show that this type of problem can be dealt with by a simple extension of the methods discussed in the preceding chapters.

The simplest and at the same time one of the most technically important problems of this type is the effect of sudden short circuits and sudden open circuits on an energized network or system. This type of problem will serve as an introduction to the more general theory.

The Sudden Short Circuit

Consider the network shown in Fig. 31.

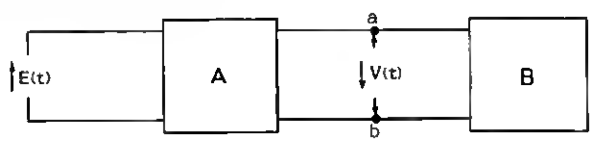

Fig. 31

This network, which for generality is supposed to consist of two parts A and B, indicated schematically, is energized by an electromotive force $E(t)$ which produces a voltage $V(t)$ between the points ab. The voltage $V(t)$ is calculable by usual methods from $E(t)$ and the constants and connections of the network, supposed to be specified.

[9] The material in this chapter is largely taken from a paper by the writer on "Theory and Calculation of Variable Electrical Systems," Phys. Rev. Feb. 1921.

We now suppose that, at reference time $t=0$, a short circuit is suddenly placed across ab; and require the effect of this short circuit on the distributions of currents in the network. The solution of this problem is based on the following proposition:

The effect of the short circuit is precisely the same as the insertion at time $t=0$ of a voltage $-V(t)$, equal and opposite to $V(t)$, between points a and b.

The resultant currents in the system for $t \geq 0$ are then composed of two components:—

(1) The currents which would exist in the invariable network, in the absence of the short circuit, due to the impressed source $E(t)$. These are calculable by usual methods.

(2) The currents due to the electromotive force $V(t)$ inserted at time $t=0$, between the points a and b. These are also calculable by usual methods, since $V(t)$ is itself known from the primary distribution of currents and charges.

By the preceding analysis we have succeeded, therefore, in reducing the problem of a sudden short circuit, to the determination of the currents in an *invariable* network in response to a suddenly impressed electromotive force: that is, the problem to which the preceding chapters have been devoted.

The Sudden Open Circuit

The problem of a sudden open circuit in any part of a network can be dealt with in a precisely analogous manner, although the actual calculation of the resultant current and voltage distribution is mathematically more complicated. Consider the network shown in Fig. 32.

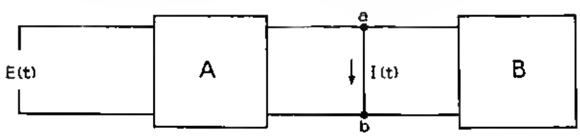

Fig. 32

Here the network is supposed to be energized by an electromotive force $E(t)$ which produced a current $I(t)$ in the *invariable* network in branch ab. We require the effect of suddenly opening this branch. The solution of this problem depends on the following proposition.

The effect of opening branch ab at reference time $t=0$ is the same as suddenly inserting at time $t=0$, a voltage $V(t)$ which produces in branch ab a current $-I(t)$ equal and opposite to the current which would exist in the branch in the absence of the open circuit.

While this proposition is precisely analogous to the corresponding proposition in the case of a sudden short circuit, it does not *explicitly* determine the voltage $V(t)$, which must be calculated as follows:

Let the driving point indicial admittance of the network, as seen from branch ab be denoted by $A_{ab}(t)$. Then, from the preceding proposition, it follows at once that $V(t)$ is given by

$$\frac{d}{dt}\int_0^t V(\tau)A_{ab}(t-\tau)d\tau = -I(t),\ t \geq 0.$$

This is a Poisson integral equation in $V(t)$, from which $V(t)$ is calculable. With $V(t)$ determined, the currents in any part of the network are calculable by usual methods, and consist of two components:—

(1) The current distribution in the network due to the impressed source $E(t)$ *in the absence of the open circuit.*

(2) The current distribution due to the electromotive force $V(t)$ inserted in branch ab at time $t=0$.

As in the case of the sudden short circuit, we have thus reduced the problem of a sudden open circuit to the determination of the current distribution in an *invariable* network in response to a suddenly impressed electromotive force.

Variable Circuit Elements

In the preceding cases of sudden open and short circuits it will be observed that the network changes discontinuously from one invariable state to another. A more general case, and one which includes the preceding as limiting cases, is presented by a network which includes a variable circuit element: that is, a circuit element which varies, continuously or discontinuously, with time. A network which includes such a variable circuit element will be called a *variable network.* Variable circuit elements of practical importance are the microphone transmitter, which consists of a variable resistance, varied by some source of energy outside the system; the condenser transmitter, which consists of a condenser of variable capacity; and the induction generator, in which the mutual inductance between primary and secondary, or stator and rotor, is varied by the motion of the latter. The case of a variable resistance will serve as an introduction to the general theory of such variable networks.

Consider a network, energized by a source $E(t)$ in branch 1, and containing a variable resistance element $r(t)$ in branch n. The functional notation $r(t)$ indicates that the resistance r varies with time. Let $I_n(t)$ denote the current in branch n, and assume that the network

is in equilibrium prior to the reference time $t=0$. The mathematical theory of this network depends on the following proposition:—

The network described above can be treated as an invariable network by eliminating the variable resistance element $r(t)$ and inserting an electromotive force $-r(t)I_n(t)$: that is, an electromotive force equal and opposite to the potential drop across the variable resistance element. Consequently the current in the variable resistance branch is determined analytically by the integral equation

$$I_n(t)=\frac{d}{dt}\int_0^t E(\tau)A_{1n}(t-\tau)d\tau-\frac{d}{dt}\int_0^t r(\tau)I_n(\tau)A_{nn}(t-\tau)d\tau. \qquad (288)$$

The first component is simply the current $I_o(t)$ which would exist in the variable branch if the variable element were absent; hence, dropping the subscript n for convenience, the current in the variable branch is given by the integral equation

$$I(t)=I_o(t)-\frac{d}{dt}\int_0^t r(\tau)I(\tau)A(t-\tau)d\tau \qquad (289)$$

and the voltage across the variable element by

$$v(t)=r(t)I(t). \qquad (290)$$

Having determined $I(t)$ and $v(t)$ from this integral equation, the distribution of currents in the network is calculable as that due to a source $E(t)$ in branch 1 and a source $v(t)$ in branch n of the *invariable network*: that is, the network with the variable resistance element eliminated.

A very simple example will serve to illustrate the foregoing:—

Into a circuit of unit resistance, and inductance $L=1/a$, in which a steady current I_o is flowing, a resistance r is suddenly inserted at time $t=0$: required the resultant current $I(t)$. In this case we have:

$$\begin{aligned}A(t)&=\text{indicial admittance of unvaried circuit}\\&=1-e^{-at},\\ r(t)&=r,\end{aligned}$$

and the integral equation of the problem is:

$$\begin{aligned}I(t)&=I_o-r\frac{d}{dt}\int_0^t(1-e^{-ay})I(t-y)dy\\&=I_o-ra\int_0^t I(t-y)e^{-ay}\,dy.\end{aligned}$$

If the solution is carried out as indicated by (291) below, and if the notation $at=x$ is introduced, we get without difficulty

$$I(t)=I_o\{1-r(1-e_1(x)e^{-x})+r^2(1-e_2(x)e^{-x})-r^3(1-e_3(x)e^{-x})+\ldots\},$$

where the function $e_n(x)$ is defined as:

$$\begin{aligned} e_n(x)&=1+x/1!+x^2/2!+x^3/3!+\ \ldots\ +x^{n-1}/(n-1)! \\ &=\text{first } n \text{ terms of the exponential series.} \end{aligned}$$

For all finite values of the resistance increment r the series can be summed by aid of the identity

$$1-e_n(x)e^{-x}=\int_0^x dxe^{-x}x^{n-1}/(n-1)!$$

Substitution of this identity gives

$$\begin{aligned} I(t)&=I_o\left(1-r\int_0^x e^{-(1+r)x}dx\right) \\ &=I_o\frac{1+re^{-(1+r)x}}{1+r}. \end{aligned}$$

Equation (289) is an integral equation of the Volterra type, which includes the Poisson integral equation as a special case. Its formal series solution is obtained as follows:—Assume a series solution of the form

$$I_n(t)=I_o(t)-I_1(t)+I_2(t)-I_3(t)+\ \ldots \tag{291}$$

and define the terms of the series by the scheme

$$\begin{aligned} I_1(t)&=\frac{d}{dt}\int_0^t r(\tau)I_o(\tau)A(t-\tau)d\tau \\ &\text{- - - - - - - - - - - - - - - - - - - -} \\ I_{k+1}(t)&=\frac{d}{dt}\int_0^t r(\tau)I_k(\tau)A(t-\tau)d\tau. \end{aligned} \tag{292}$$

Direct substitution shows that this series satisfies the integral equation. Furthermore, it is easily shown that it is absolutely convergent.

While this series solution is not, in general, well adapted for numerical calculations, it throws a good deal of valuable light on the ultimate character of the oscillations in the important case where $E(t)$ and $r(t)$ both vary sinusoidally with time. In this case, if the frequency of the applied e.m.f. be denoted by F and that of the resistance variation by f, it is easy to show that the current $I_o(t)$ in the unvaried

circuit is ultimately [10] a steady state current of frequency F. This follows from the fact that the definite integral which defines the current $I_o(t)$ is resolvable into the ultimate steady state current corresponding to an applied force of frequency F, and the accompanying transient oscillations which ultimately die away. The fictitious e.m.f. which may be regarded as producing the component current $I_1(t)$ is $rf(t)I_o(t)$; this is ultimately the product of the two frequencies F and f, and therefore resolvable into two terms of frequency $F+f$ and $F-f$ respectively. Carrying through this analysis, it is easy to show that each component current is ultimately a steady-state but poly-periodic oscillation, as indicated in the following table:

Component Current	Frequency
I_0	$F,$
I_1	$F+f, F-f,$
I_2	$F+2f, F, F-2f,$
I_3	$F+3f, F+f, F-f, F-3f,$
I_4	$F+4f, F+2f, F, F-2f, F-4f.$

It is of importance to observe that the component currents involve, from a mathematical standpoint, multiple integrals of successively higher orders, the nth component $I_n(t)$ involving a multiple integral of the n^{th} order with respect to $I_o(t)$. Consequently the successive currents require longer and longer intervals of time to build up to their proximate steady-state values, so that the time required for the resultant steady-state to be reached cannot be inferred from the time constant of the unvaried circuit.

From the preceding table it will be seen that the ultimate steady-state current is obtained by rearranging the series $I_o+I_1+I_2$ and is of the form

$$\sum_{n=-\infty}^{+\infty} A_n \cos\,(\Omega+n\omega)t+B_n \sin\,(\Omega+n\omega)t$$

where $\Omega=2\pi F$ and $\omega=2\pi f$.

It is interesting to note that this series comes within the definition of a Fourier series only when $F=0$ or an exact multiple of f. The steady-state solution is of very considerable importance and is considered in more detail in a succeeding chapter.

From the foregoing we deduce an outstanding distinction between the variable and invariable networks. In the latter the currents are

[10] It hardly seems necessary to remark that the reference time $t=o$ is purely arbitrary and that the resistance variation may start at such a time thereafter that $I_o(t)$ may be regarded as steady state during the entire time interval in which we are interested. Going farther, if we confine our attention to sufficiently large values of t, the whole process may be treated as steady state.

ultimately of the same frequency as the impressed e.m.f., whereas in the former they are ultimately of an infinite series of frequencies.

In the preceding example, the variable impedance element is a resistance $r(t)$. If the variable element is taken as an *inductance* $\lambda(t)$ the voltage, corresponding to equation (290) is

$$\frac{d}{dt}\lambda(t)I(t).$$

The case of a variable capacity element is handled as follows: Let $1/C=S$ and assume that S is variable: thus, $S=S_o+\sigma(t)$. The drop across the variable condenser element is then

$$v(t)=\sigma(t)\int_0^t I(t)dt.$$

Similarly a variable mutual inductance $\mu(t)$ between branches m and n produces the voltages

$$\frac{d}{dt}\mu(t)I_n(t)$$

in branch m, and

$$\frac{d}{dt}\mu(t)I_m(t)$$

in branch n. This case may be illustrated by:

The Induction Generator Problem

In a sufficiently general form, this problem, which includes the fundamental theory of the dynamo, may be stated as follows:

Given an invariable primary and secondary circuit with a variable mutual inductance $Mf(t)$ which is an arbitrary but specified time function, and let the primary be energized by an e.m.f. $E(t)$ impressed in the circuit at the reference time $t=0$: required the primary and secondary currents.

In operational notation the problem may be formulated by the equations:

$$Z_{11}I_1-pMf(t)I_2=E(t),$$

$$-pMf(t)I_1+Z_{22}I_2=0,$$

in which Z_{11} and Z_{22} are the self impedances of the primary and secondary respectively; $Mf(t)$ is the variable mutual inductance; $E(t)$ is the applied e.m.f. in the primary; and p denotes the differential

operator d/dt. By aid of the fundamental formula these equations may be written down as the following simultaneous integral equations:

$$I_1(t)=\frac{d}{dt}\int_0^t dyA_{11}(t-y)\left(E(y)+M\frac{d}{dy}[f(y)I_2(y)]\right)$$

$$I_2(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)I_1(y)].$$

In these equations, $A_{11}(t)$ and $A_{22}(t)$ denote the indicial admittances of the primary and secondary circuits respectively (when $M=0$): that is, the currents in these circuits in response to a unit e.m.f. (zero before, unity after time $t=0$). We assume, of course, that they are known or can be determined by usual methods.

It follows at once that the formal solution of these equations is the infinite series:

$$I_1(t)=X_o(t)+X_2(t)+X_4(t)+\ \dots\ +X_{2n}(t)+\ \dots$$

$$I_2(t)=Y_1(t)+Y_3(t)+Y_5(t)+\ \dots$$

in which the successive terms of the series are defined as follows:

$$X_o(t)=\frac{d}{dt}\int_0^t dyA_{11}(t-y)E(y)=I_o(t),$$

$$Y_1(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_o(y)],$$

$$X_2(t)=M\frac{d}{dt}\int_0^t dyA_{11}(t-y)\frac{d}{dy}[f(y)Y_1(y)],$$

$$Y_3(t)=M\frac{d}{dt}\int_0^t dyA_{22}(t-y)\frac{d}{dy}[f(y)X_2(y)],\quad \text{etc.}$$

In the light of formula

$$I(t)=\frac{d}{dt}\int_0^t f(y)A(t-y)dy$$

the physical interpretation of the series solutions follows at once: Thus, $X_o(t)$ is equal to the current $I_o(t)$ flowing in the *isolated* primary in response to the applied e.m.f. $E(t)$; the first component current $Y_1(t)$ in the secondary is equal to the current which would flow in the isolated secondary in response to the applied e.m.f. $M(d/dt)f(t)X_o(t)$; $X_2(t)$, the second component current in the primary, is equal to the current in the isolated primary in response to the applied e.m.f. $M(d/dt)$

$f(t)Y_1(t)$; etc. The resultant currents are thus represented as built up by a to-and-fro interchange of energy between primary and secondary, or by a series of successive reactions. In the important case where the applied e.m.f. and the variation of mutual inductance are both sinusoidal time functions, of frequency F and f respectively, it is easy to show that each component current becomes ultimately equal to a set of periodic steady-state currents. Thus the component X_o is ultimately single periodic, of frequency F; Y_1 is ultimately doubly periodic, of frequencies $F+f$ and $F-f$; X_2 triply periodic, of frequencies $F+2f$, F and $F-2f$; Y_3 quadruply periodic, of frequencies $F+3f$, $F+f, F-f, F-3f$; etc.

The Solution for the Steady-State Oscillations

For the very important case of periodic applied forces and periodic variations of circuit elements we are often concerned exclusively with the ultimate steady-state of the system, and not at all with the mode in which the steady-state is approached: that is, attention is restricted to the periodic oscillations which the system executes after transient disturbances have died away. In this case, if the periodic variations of circuit elements are sufficiently small, the required steady-state is obtained in the form of a series by replacing each term of the complete series solution by its ultimate steady-state value; a process which is very simple in view of the physical significance of each term of the latter series. The appropriate procedure will be briefly illustrated in connection with the variable resistance element. In view of the fact that we are concerned only with the ultimate steady-state oscillations, we can base the solutions on the symbolic equation

$$I = I_o - \frac{r(t)}{Z} I. \tag{293}$$

Here $r(t)$ is the variable resistance element; I_o is the current which would flow in the absence of the resistance variation; and Z is a generalized impedance of the network, as seen from the variable branch. Its precise significance and functional form is given below.

We now suppose that I_o is given by

$$I_o = J_o e^{i\Omega t} \qquad \text{(real part)} \tag{294}$$

$$= \tfrac{1}{2}\left(J_o e^{i\Omega t} + \overline{J}_o e^{-i\Omega t}\right) \tag{295}$$

where the bar indicates the conjugate imaginary of the unbarred

symbol, so that (295) is entirely real. Correspondingly the variable resistance will be taken as

$$r(t) = \frac{r}{2}(e^{i\omega t}+e^{-i\omega t})$$

$$= re^{i\omega t} \qquad \text{(real part)} \tag{296}$$

$$= r \cos \omega t.$$

Here r is taken as a pure real quantity, which fixes the size of the resistance variation. No loss of generality is involved in this, since it merely involves referring the time scale to the zero of the resistance variation.

The symbolic impedance Z, as employed in the theory of alternating currents, will depend on the frequency and is, in general, a complex quantity. Its value at frequency $\Omega/2\pi$ will be denoted by

$$Z(i\Omega) = Z_o$$

while its value at frequency $(\Omega+n\omega)/2\pi$ will be written as

$$Z(i(\Omega+n\omega)) = Z_n.$$

We now assume a series solution of (293) of the form

$$I = I_o+I_1+I_2+ \ . \ . \ .$$

where the terms of the series are defined by the symbolic equations

$$I_1 = -\frac{r(t)}{Z} I_o,$$

$$- \ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \tag{297}$$

$$I_{n+1} = -\frac{r(t)}{Z} I_n.$$

Substitution shows that this series formally satisfies the equation.

Starting with the first of (297) and substituting (295) and (296) we get

$$I_1 = -\frac{r}{4Z}(e^{i\omega t}+e^{-i\omega t})\ (J_o e^{i\Omega t}+\bar{J}_o e^{-i\Omega t})$$

$$= -\frac{r}{4Z}\left\{ J_o e^{i(\Omega+\omega)t}+\bar{J}_o e^{-i(\Omega+\omega)t}+J_o e^{i(\Omega-\omega)t}+\bar{J}_o e^{-i(\Omega-\omega)t} \right\}, \tag{298}$$

or

$$I_1 = -\frac{r}{2}J_o\left\{ \frac{e^{i(\Omega+\omega)t}}{Z_1}+\frac{e^{i(\Omega-\omega)t}}{Z_{-1}} \right\}. \tag{299}$$

In (299) it is to be understood that the real part is alone to be retained.

Proceeding in a similar way with the equation

$$I_2 = -\frac{r(t)}{Z} I_1$$

we get

$$I_2 = \left(\frac{r}{2}\right)^2 J_o \left\{ \frac{e^{i(\Omega+2\omega)t}}{Z_1 Z_2} + \frac{e^{i(\Omega-2\omega)t}}{Z_{-1} Z_{-2}} + \frac{e^{i\Omega t}}{Z_o}\left(\frac{1}{Z_1} + \frac{1}{Z_{-1}}\right) \right\}. \tag{300}$$

In this way the steady-state series solution is built up term by term, the component currents being poly-periodic as indicated in a previous table.

For sufficiently small impedance variations this method of solution works very well, and leads to a rapidly convergent solution. In other cases, however, the solution so obtained may be divergent, even when the complete series solution from which it is derived is absolutely convergent. The explanation of this lies in the fact that the steady-state series so obtained is the *sum of the limits* (as t approaches infinity) of the terms of the complete series solution, whereas the actual steady-state is the *limit of the sum*. These are not in general equal; in particular the former may be and often is divergent when the latter is convergent.

In view of the foregoing considerations it is of importance to develop another method of investigating the steady-state oscillations which avoids the difficulties in the formal series solution. The following method has suggested itself to the writer and works very well in cases where the previous form of solution fails. It should be stated at the outset, however, that the absolute convergence of the solution to be discussed, while reasonably certain in all physically possible systems, has not been established by a rigorous mathematical investigation, which appears to present very considerably difficulties.

We start with the problem just discussed and, in view of the results of the formal series solution there obtained, assume a solution of the form:

$$I = \frac{1}{2}\sum_{-N}^{N} A_m e^{i(\Omega+m\omega)t} + \bar{A}_m e^{-i(\Omega+m\omega)t} \tag{301}$$

$$= \sum_{-N}^{N} A_m e^{\,i(\Omega+m\omega)t} \qquad \text{(real part)}. \tag{302}$$

Here the series is supposed to extend from $m = +N$ to $m = -N$. Ultimately, however, N will be put equal to infinity. As before, the

bar indicates the conjugate imaginary of the unbarred symbol and (301) is therefore entirely real.

If we now substitute (301) in the symbolic equation (293) we get, by (295) and (296),

$$\frac{1}{2}\sum\left\{A_m e^{i(\Omega+m\omega)t}+\bar{A}_m e^{-i(\Omega+m\omega)t}\right\}=\tfrac{1}{2}J_o e^{i\Omega t}+\tfrac{1}{2}\bar{J}_o e^{-i\Omega t}$$
$$-\frac{r}{2Z}\left(e^{i\omega t}+e^{-i\omega t}\right)\sum\left\{A_m e^{i(\Omega+m\omega)t}+\bar{A}_m e^{-i(\Omega+m\omega)t}\right\}.$$

Simplifying this equation and dropping the conjugate imaginaries gives:—

$$\sum A_m e^{i(\Omega+m\omega)t}=J_o e^{i\Omega t}-\frac{r}{Z}\sum A_m e^{i(\Omega+(m+1)\omega)t} \tag{303}$$
$$-\frac{r}{Z}\sum A_m e^{i(\Omega+(m-1)\omega)t}.$$

Finally, if we write

$$Z(i(\Omega+m\omega))=Z_m$$

and

$$r/Z_m=h_m \tag{304}$$

and equate terms of the same frequency on the two sides of the equation, we get

$$\left.\begin{aligned}
A_N&=-h_N A_{N-1}\\
A_m&=-h_m(A_{m-1}+A_{m+1}) \qquad 0<|m|<N\\
A_o&=J_o-h_o(A_{-1}+A_1).
\end{aligned}\right\} \tag{305}$$

It will be observed that, by (305), starting with A_N each coefficient is determined in terms of the coefficient of the next lower index. Thus:

$$\begin{aligned}
A_N&=-h_N A_{N-1}\\
A_{N-1}&=-h_{N-1}(A_{N-2}+A_N)\\
&=-\frac{h_{N-1}A_{N-2}}{1-h_{N-1}h_N}.
\end{aligned}$$

Similarly

$$A_{N-2}=-\cfrac{h_{N-2}A_{N-3}}{1-h_{N-2}h_{N-1}\cfrac{1}{1-h_{N-1}h_N}}.$$

Continuing this process it is easy to show that, for positive indices (m positive),

$$A_m = -h_m C_m A_{m-1} \tag{306}$$

where C_m designates the continued fraction

$$\frac{1}{1-h_m h_{m+1}}\;\frac{1}{1-h_{m+1}h_{m+2}}\;\cdots\;\frac{1}{1-h_{N-1}h_N}$$

The procedure for the coefficient A_{-m} is precisely similar. For convenience we write $A_{-m}=A'_m$, $Z_{-m}=Z'_m$, and $r/Z_{-m}=h'_m$. In this notation we get by precisely similar procedure

$$A'_m = -h'_m C'_m A'_{m-1} \tag{307}$$

where C'_m designates the continued fraction

$$\frac{1}{1-h'_m h'_{m+1}}\;\frac{1}{1-h'_{m+1}h'_{m+2}}\;\cdots\;\frac{1}{1-h'_{N-1}h'_N}$$

We now put the index N equal to infinity and the continued fractions C_m and C'_m become infinite instead of terminating fractions.

Collecting formulas we now have

$$A_m = -h_m C_m A_{m-1}$$

$$A'_m = -h'_m C'_m A'_{m-1}$$

and

$$A_o = J_o - (h_o A_1 + h'_o A'_1)$$

whence

$$A_o = \frac{J_o}{1-h_o h_1 C_1 - h_o' h_1' C_1'}.$$

The coefficients are thus all determined in terms of J_o.

The practical value of this method of solution will depend, of course, on the rate of convergence of the continued fractions. While no rigorous proof has been obtained, it is believed that they are absolutely convergent for all physically possible systems, but this question certainly requires fuller investigation. Nevertheless any doubt regard-

ing the convergence of the solution need not prevent the use of the method in a great many problems where physical considerations furnish a safe guide. For example this method of solution, when applied to the problem of the induction generator, discussed above, leads to the usual simplified engineering theory of the induction generator and motor, besides exhibiting effects which the usual treatment either ignores or fails to recognize.

Non-Linear Circuits

In the previous examples discussed, the variations of the variable circuit elements are assumed to be specified time functions, which is the same thing as postulating that these variations are controlled by ignored forces which do not explicitly appear in the statement and equations of the problem. We distinguish another type of variable circuit element, where the variation is not an explicit time function but rather a function of the current (and its derivatives) which is flowing through the circuit. For example, the inductance of an iron-core coil varies with the current strength as a consequence of magnetic saturation. The equation of a circuit which contains such a variable element (provided it is a single valued function) may be written down in operational notation

$$ZI+\phi(I)=E(t),$$

or

$$ZI=E(t)-\phi[I(t)]. \tag{311}$$

In this equation Z is, of course, to be taken as the impedance of the invariable part of the circuit, the indicial admittance of which is denoted by the usual symbol $A(t)$.

Equation (311) may be interpreted as the equation of the current $I(t)$ in a circuit of invariable impedance Z when subjected to an applied e.m.f. $E(t)-\phi[I(t)]$; consequently, by aid of our fundamental formula, $I(t)$ is given by

$$I(t)=\frac{d}{dt}\int_0^t A(t-y)E(y)dy-\frac{d}{dt}\int_0^t A(t-y)\phi[I(y)])dy.$$

The first integral is simply the current in the invariable circuit of impedance Z in response to the applied e.m.f. $E(t)$; denoting this by $I(t)$, we have

$$I(t)=I_0(t)-\frac{d}{dt}\int_0^t A(t-y)\phi[I(y)]dy.$$

This is a *functional integral equation*, the solution of which is gotten

by some process of successive approximations. For example, provided the sequence converges, $I(t)$ is the limit as n approaches infinity of the *sequence*

$$I_o(t),\ I_1(t),\ I_2(t),\ \ldots,\ I_n(t),$$

where the successive terms of the sequence are defined by the relations:

$$I_1(t)=I_o(t)-\frac{d}{dt}\int_0^t A(t-y)\phi[I_o(y)]dy,$$

$$- -$$

$$I_{n+1}(t)=I_o(t)-\frac{d}{dt}\int_0^t A(t-y)\phi[I_n(y)]dy.$$

We shall not pursue the discussion of non-linear circuits further, in view of their mathematical complexity and their relatively specialized technical interest. The reader who is interested may, however, consult the writer's paper on Variable Electrical Systems,[11] for a fuller treatment of the subject.

CHAPTER XI

The Application of the Fourier Integral to Electric Circuit Theory

The application of Fourier's series in electrotechnics is a commonplace; the use of the Fourier integral, however, has largely remained in the hands of professional mathematicians. An outstanding distinction between the series and the integral, from which the greater power of the latter may be inferred, is that the series represents only a periodic regularly recurrent function, whereas the integral is capable of representing a non-periodic function: in fact all types of functions, subject to certain mathematical restrictions which are usually satisfied in physical problems.

Before taking up the application of the Fourier Integral to Electric Circuit Theory, we shall very briefly review the elementary mathematics of the series and integral; for a fuller treatment the reader is referred to Byerly, Fourier's Series and Spherical Harmonics.[12]

Consider a function $\phi(t)$, which in the region $0 \leq t \leq T$ is finite, single-

[11] Phys. Rev. Feb., 1921.

[12] In this chapter the Fourier Integral is approached from the view-point of its physical application and no completeness or rigour is claimed for the treatment. The mathematical theory of the Fourier integral is, of course, completely developed in treatises on the subject. The object of this chapter is merely to outline some of its applications.

valued and has only a finite number of discontinuities or of maxima or minima. In this region it can then be expressed as the Fourier series

$$\phi(t)=\tfrac{1}{2}A_0+\sum_1^\infty\left\{A_n\cos\left(\frac{2\pi n}{T}t\right)+B_n\sin\left(\frac{2\pi n}{T}t\right)\right\} \tag{312}$$

where

$$\begin{aligned}A_n&=\frac{2}{T}\int_0^T\phi(t)\cdot\cos\left(\frac{2\pi n}{T}t\right)dt,\\ B_n&=\frac{2}{T}\int_0^T\phi(t)\cdot\sin\left(\frac{2\pi n}{T}t\right)dt.\end{aligned} \tag{313}$$

An equivalent series is

$$\phi(t)=\tfrac{1}{2}F_o+\sum_1^\infty F_n\cos\left(\frac{2\pi n}{T}t-\Theta_n\right) \tag{314}$$

where

$$\begin{aligned}F_n&=\sqrt{A_n^2+B_n^2},\\ \Theta_n&=\tan^{-1}(B_n/A_n).\end{aligned} \tag{315}$$

This expansion is valid in the region $0\leq t\leq T$, irrespective of the form of the function elsewhere. Let us, however, assume that the function repeats itself in the period T: that is

$$\phi(t\pm kT)=\phi(t),\qquad k=1,\ 2,\ 3\ \ldots\ N.$$

Then the expansion represents the function in the region $-NT\leq t\leq NT$. Finally if N is made infinite, the function is truly periodic and the Fourier series represents it for all positive and negative values of time.

It follows from the foregoing that, if the Fourier series represents the function for all positive and negative values of time, the function must be periodic for all positive and negative values of time; otherwise the expansion is valid only over a restricted range of time.

Now let us suppose that $\phi(t)$ is non-periodic. For convenience, in connection with subsequent applications we shall suppose that it is zero for all finite *negative* values of time, that it converges to zero as $t\rightarrow\infty$, and that

$$\int_0^\infty \phi(t)\ \ dt$$

exists. Such a function obviously cannot be represented by the usual Fourier series for all finite positive and negative values of time; it

can be represented, however, by the limiting form assumed by the series as the fundamental period T is made infinite. That is, we can assume that the function is periodic in an infinite fundamental period and this will not affect the expansion for finite positive and negative values of time. Proceeding in this way and putting the fundamental period T equal to infinity in the limit, the Fourier series (314) becomes an infinite integral and we get

$$\phi(t) = \frac{1}{\pi} \int_0^{\infty} F(\omega) \cdot \cos\left(\omega t - \theta(\omega)\right) d\omega \tag{316}$$

where

$$F(\omega) = \left\{ \left[\int_0^{\infty} \phi(t) \cos \omega t \, dt \right]^2 + \left[\int_0^{\infty} \phi(t) \sin \omega t \, dt \right]^2 \right\}^{\frac{1}{2}} \tag{317}$$

and

$$\tan \theta(\omega) = \int_0^{\infty} \phi(t) \sin \omega t \, dt \div \int_0^{\infty} \phi(t) \cos \omega t \, dt. \tag{318}$$

This is the *Fourier integral* identity of the function $\phi(t)$ and is valid for all finite positive and negative values of time.

In physical applications, particularly those to electric circuit theory, it is often convenient to employ exponential instead of trigonometric functions. The required transformation follows easily from the relation

$$e^{i\theta} = \cos \theta + i \sin \theta, \qquad i = \sqrt{-1}.$$

Thus if we write $2\pi/T = \omega_o$ the Fourier series (312) is easily reduced to the form

$$\phi(t) = \sum_{-\infty}^{+\infty} F(in\omega_o) e^{in\omega_o t} \tag{319}$$

where

$$F(in\omega_o) = \frac{1}{T} \int_0^{T} \phi(\tau) \, e^{-in\omega_o \tau} \, d\tau. \tag{320}$$

In precisely similar manner the Fourier integral (316) can be written as

$$\phi(t) = \int_{-\infty}^{\infty} F(i\omega) \cdot e^{i\omega t} d\omega \tag{321}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \phi(\tau) d\tau \int_{-\infty}^{\infty} e^{i\omega(t-\tau)} d\omega. \tag{322}$$

Applications to Electric Circuit Theory

Let us assume that at time $t=-NT$, an electromotive force $E(t)$, periodic in fundamental period T, is impressed on a circuit of complex impedance $Z(i\omega)$, where ω denotes 2π times the frequency. Required the resultant current I.

For values of $t>-NT$ the electromotive force (see formula (319)) can be expressed as the Fourier series

$$E(t)=\sum_{-\infty}^{\infty}F(in\omega_o)e^{in\omega_o t}$$

where

$$F(in\omega_o)=\frac{1}{T}\int_0^T E(\tau)e^{-in\omega_o\tau}d\tau.$$

The resultant current for $t>-NT$ is therefore

$$I=\sum_{-\infty}^{\infty}\frac{F(in\omega_o)}{Z(in\omega_o)}e^{in\omega_o t}+\left\{\begin{array}{l}\text{transient oscillations}\\ \text{initiated at time}\\ t=-NT.\end{array}\right\}.$$

If we are concerned with the current for values of $t\geq 0$, and if NT is made sufficiently large, the initial transients will have died away and the *complete current* for $t\geq 0$, will be given by

$$I=\sum_{-\infty}^{\infty}\frac{F(in\omega_o)}{Z(in\omega_o)}e^{in\omega_o t}. \tag{323}$$

This formula implies the periodic character of $E(t)$ for sufficiently large negative values of time. If, however, $E(t)$ is zero for negative values of time, we can employ the Fourier integrals (321) and (322) in precisely the same way and get, as the *complete* expression for the current for positive or negative values of time:—

$$I=\int_{-\infty}^{\infty}\frac{F(i\omega)}{Z(i\omega)}e^{i\omega t}\,d\omega \tag{324}$$

$$=\frac{1}{2\pi}\int_0^{\infty}E(\tau)d\tau\int_{-\infty}^{\infty}\frac{e^{i\omega(t-\tau)}}{Z(i\omega)}\,d\omega. \tag{325}$$

The infinite integrals (324) *and* (325) *formulate the current in the network, specified by the impedance function* $Z(i\omega)$, *in response to an electromotive force* $E(t)$ *impressed at time* $t=0$; *they therefore mathematically formulate, by aid of the Fourier integral identity, the fundamental problem dealt with in the preceding chapters and solved by aid of the operational calculus.*

No attempt will be made here to discuss the solution of the infinite integral (325), which is usually a problem presenting formidable difficulties, even to the professional mathematician. The general method of solution is by contour integration in the complex plane and the calculus of residues. By this process it has been successfully applied to the solution of special problems, and also to deriving some general forms of solution such as the expansion theorem solution.[13] Compared, however, with the operational calculus, it has no advantages from the standpoint of rigour, and lacks entirely the remarkable simplicity and directness of the Heaviside method.

In the direct solution of circuit problems, therefore, it is believed that the application of the Fourier integral is attended by few if any advantages, and presents formidable mathematical difficulties. On the other hand, there are certain types of problems encountered in circuit theory, where the Fourier integral is a powerful tool. These will be briefly discussed.

The Energy Absorbed from Transient Applied Forces

In many technical problems, the complete solution for the instantaneous current due to suddenly applied electromotive forces, although formally straight-forward, involves a prohibitive amount of labor. In yet others, the applied forces may be random and specified only by their mean square values. In such problems a great deal of useful information is furnished by the mean power and mean square current absorbed by the network, and to the calculation of these quantities, the Fourier integral is ideally adapted. Its application depends on the following proposition, due to Rayleigh (Phil. Mag., Vol. 27, 1889, p. 466), and its corollary.

Let a function $\phi(t)$, supposed to exist only in the epoch $0 \leq t \leq T$, be formulated as the Fourier integral

$$\phi(t) = \frac{1}{\pi}\int_0^\infty |f(\omega)| \cdot \cos\,[\omega t - \theta(\omega)]\,d\omega$$

where

$$f(\omega) = \left\{\left[\int_0^T \phi(t)\cos\omega t\,dt\right]^2 + \left[\int_0^T \phi(t)\sin\omega t\,dt\right]^2\right\}^{\frac{1}{2}}$$

$$\tan\theta(\omega) = \int_0^T \phi(t)\sin\omega t\,dt \div \int_0^T \phi(t)\cos\omega t\,dt.$$

[13] Bush, "Summary of Wagner's Proof of Heaviside's Formula." Proc. Inst. of Radio Engineers. Oct., 1917. Fry. "The Solution of Circuit Problems." (Phys. Rev. Aug., 1919).

Then

$$\int_0^T [\phi(t)]^2\, dt = \frac{1}{\pi}\int_0^\infty |f(\omega)|^2\, d\omega,$$

whereby the time integral is transformed into an integral with respect to frequency.

A corollary of this theorem is as follows:

If two functions $\phi_1(t)$, $\phi_2(t)$ supposed to exist only in the epoch $0 \leq t \leq T$, are formulated by the Fourier integrals

$$\phi_1(t) = \frac{1}{\pi}\int_0^\infty |f_1(\omega)| \cdot \cos\,[\omega t - \theta_1(\omega)]\, d\omega,$$

$$\phi_2(t) = \frac{1}{\pi}\int_0^\infty |f_2(\omega)| \cdot \cos\,[\omega t - \theta_2(\omega)]\, d\omega,$$

then

$$\int_0^T \phi_1(t)\phi_2(t) dt = \frac{1}{\pi}\int_0^\infty |f_1(\omega)| \,.\, |f_2(\omega)| \,.\, \cos(\theta_1 - \theta_2) d\omega.$$

The applications of these theorems to circuit theory proceeds as follows:—

If an electromotive force $E(t)$, supposed to exist only in the epoch $0 \leq t \leq T$, is applied to a network of complex impedance $Z(i\omega) = |Z(i\omega)|\, e^{i\beta(\omega)}$ we know from the preceding discussion of the Fourier integral, that the electromotive force $E(t)$ and current $I(t)$ are expressible as the Fourier integrals

$$\begin{aligned} E(t) &= \frac{1}{\pi}\int_0^\infty |f(\omega)| \,.\, \cos\,(\omega t - \theta(\omega)) d\omega, \\ I(t) &= \frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|}{|Z(i\omega)|} \cos(\omega t - \theta(\omega) - \beta(\omega)) d\omega. \end{aligned} \tag{326}$$

It follows at once from Rayleigh's theorem that

$$\int_0^\infty I^2 dt = \frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|^2}{|Z(i\omega)|^2} d\omega. \tag{327}$$

Now let I_n be the current absorbed in branch n; let $z(i\omega) = |z(i\omega)|\, e^{i\alpha(\omega)}$ be the impedance of that branch and let $E_n(t)$ be the potential drop across that branch. It follows at once from the corollary to Rayleigh's theorem that

$$W = \int_0^\infty E_n(t) I_n(t) dt = \frac{1}{\pi}\int_0^\infty \frac{|f(\omega)|^2}{|Z(i\omega)|^2} |z(i\omega)| \cos \alpha(\omega) d\omega. \tag{328}$$

Formulas (327) and (328) formulate the mean square current and mean power absorbed by the branch of the network under consideration, and enable us to calculate these quantities, even in the case of complicated networks, with a minimum of labor. Formula (327) is particularly well adapted to computation because the integrand is everywhere positive, permitting, in most problems, of easy numerical integration, whereas the analytical solution may be complicated.

Formulas (327) and (328) have been applied to the theory of selective circuits, to the problem of interference from random disturbances, including static, and to the theory of the Schrotteffekt. For the details of such applications, which will not be entered into here, the reader is referred to the following papers.

Transient Oscillations in Electric Wave Filters, Bell System Technical Journal, July, 1923.

Selective Circuits and Static Interference, Trans. A. I. E. E., 1924.

An Application of the Periodogram to Wireless (Burch & Bloehmsma), Phil. Mag., Feb., 1925.

The Theory of the Schrotteffekt (Fry), Journal Franklin Institute, Feb., 1925.

The Building-Up of Alternating Currents

Another application of the Fourier Integral, which may be briefly mentioned, is to the building-up of alternating currents in response to suddenly impressed sinusoidal electromotive forces. The investigation of this problem is of great importance to the communication engineer, since the excellence of a signal transmission system is to a considerable extent determined by the duration and character of the building-up phenomena.

In long transmission systems the calculation of the building-up current as a time function is extremely complicated and laborious if not practically impossible. Furthermore we are usually not concerned with the current as an instantaneous time function, but rather with its *envelope*. The envelope of the current can be formulated and calculated by modified Fourier integrals, by the following process.

Suppose that an e.m.f. $E \cos \omega t$ is suddenly applied, at reference time $t=0$, to a network of transfer impedance

$$Z(i\omega)=|Z(i\omega)|\, e^{iB(\omega)}.$$

The resultant current $I(t)$ may always be written as:

$$I(t)=\frac{1}{2}\frac{E}{|Z(i\omega)|}\left\{(1+\rho)\cos(\omega t-B)+\sigma\sin(\omega t-B)\right\}$$

$$=\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}\frac{E}{|Z(i\omega)|}\cos(\omega t-B(\omega)-\theta)$$

where

$$\theta=\tan^{-1}\frac{\sigma}{1+\rho}$$

Evidently the functions ρ and σ, which it is our problem to determine, must be -1 and 0 respectively for negative values of t, and approach the limits $+1$ and 0, respectively, as $t\to\infty$.

In an engineering study of the building-up process we are principally concerned with the *envelope* of the oscillations: hence with

$$\frac{1}{2}\sqrt{(1+\rho)^2+\sigma^2}.$$

Our problem is therefore to determine the functions ρ and σ and to examine the effect of the applied frequency $\omega/2\pi$ and of the characteristics of the circuit, on their rate of building-up and mode of approach to their ultimate steady values.

The functions ρ and σ can be formulated as the Fourier integrals

$$\rho=\frac{1}{\pi}\int_0^\infty[P_\omega(\lambda)+P_\omega(-\lambda)]\sin t\lambda.\frac{d\lambda}{\lambda}$$

$$-\frac{1}{\pi}\int_0^\infty[Q_\omega(\lambda)-Q_\omega(-\lambda)]\cos t\lambda\frac{d\lambda}{\lambda}$$

$$\sigma=\frac{1}{\pi}\int_0^\infty[Q_\omega(\lambda)+Q_\omega(-\lambda)]\sin t\lambda\frac{d\lambda}{\lambda}$$

$$+\frac{1}{\pi}\int_0^\infty[P_\omega(\lambda)-P_\omega(-\lambda)]\cos t\lambda\frac{d\lambda}{\lambda},$$

where

$$P_\omega(\lambda)=\frac{|Z(i\omega)|}{|Z(i\omega)+i\lambda)|}\cdot\cos[B(\omega+\lambda)-B(\omega)],$$

$$Q_\omega(\lambda)=\frac{|Z(i\omega)|}{Z(i\omega+i\lambda)|}\cdot\sin[B(\omega+\lambda)-B(\omega)].$$

These formulas are directly deducible from the fact that the applied e.m.f., defined as zero for $t<0$ and $E \cos \omega t$ for $t \geq 0$, can itself be expressed as

$$\frac{E}{2}\cos \omega t\left[1+\frac{2}{\pi}\int_0^\infty \sin t\lambda \frac{d\lambda}{\lambda}\right].$$

For important types of transmission systems, including the periodically loaded line, these formulas have been successfully dealt with and solutions of a satisfactory approximate character obtained. For further details, the reader is referred to a paper on "The Building-Up of Sinusoidal Currents in Long Periodically Loaded Lines" (Bell System Technical Journal, October, 1924).

The foregoing must conclude our very brief account of the Fourier Integral and its applications in Electric Circuit theory; an adequate treatment of this subject would require a treatise in itself, and is beyond the scope of the present work. All that has been attempted is to give a very brief introduction to its significance in physical problems and a few of its outstanding applications in circuit theory. The reader who is interested in pursuing this subject further is referred to a paper by T. C. Fry on "The Solution of Circuit Problems" (Phys. Rev., Aug., 1919), which gives a rigorous discussion of the solution of the Fourier Integral by contour integration, together with some general forms of solution of the circuit problem.[1]

[1] It was planned to include in this paper a bibliography of the important papers bearing on the Heaviside operational method. This, however, has not been completed, but plans call for its publication in the next issue.—Editor.